

Isolated spectral points and Koliha-Drazin invertible elements in quotient Banach algebras and homomorphism ranges

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Abstract

In this article poles, isolated spectral points, group, Drazin and Koliha-Drazin invertible elements in the context of quotient Banach algebras or in ranges of (not necessarily continuous) homomorphism between complex unital Banach algebras will be characterized using Fredholm and Riesz Banach algebra elements. Calkin algebras on Banach and Hilbert spaces will be also considered.

Keywords: Isolated spectral point, pole, Koliha-Drazin inverse, Drazin inverse, Banach algebra, Calkin algebra.

1. Introduction

In the article [6], given a separable infinite dimensional Hilbert space \mathcal{H} and $T \in \mathcal{L}(\mathcal{H})$, it was characterized when zero is a pole of the resolvent of $\pi(T) \in \mathcal{C}(\mathcal{H})$ in terms of the operator $T \in \mathcal{L}(\mathcal{H})$, where $\mathcal{C}(\mathcal{H})$ is the Calkin algebra and $\pi: \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{C}(\mathcal{H})$ is the quotient map. In fact, zero is a pole of the resolvent of $\pi(T) \in \mathcal{C}(\mathcal{H})$ if and only if there are operators A , B and $K \in \mathcal{L}(\mathcal{H})$ such that A is nilpotent, B is Fredholm, K is compact and $T = A \oplus B + K$ (see [6, Theorem 2.2]). It is worth noticing that according to [32, Proposition 1.5] or [10, Theorem 12], given a complex unital Banach algebra \mathcal{A} , the set of the poles of the resolvent of $a \in \mathcal{A}$ coincide with $\{\lambda \in \text{iso } \sigma(a): \text{ such that } a - \lambda 1 \text{ is Drazin invertible}\}$, where $\text{iso } \sigma(a)$ denotes the set of isolated points of the spectrum $\sigma(a)$ and $1 \in \mathcal{A}$ denotes the identity of \mathcal{A} . Furthermore, according to [32, Proposition 1.5], $\text{iso } \sigma(a) = \{\lambda \in \sigma(a): \text{ such that } a - \lambda 1 \text{ is Koliha-Drazin invertible}\}$.

On the other hand, recall that according to Atkinson's theorem, necessary and sufficient for a bounded linear operator on a Banach space to be Fredholm is that its coset in the Calkin algebra is invertible. Motivated for this result, an abstract Fredholm theory in arbitrary complex unital Banach algebras with respect to an inessential ideal was developed, see for example [1, 5] and the bibliography in these monographs. In addition, a Fredholm theory relative to Banach algebra homomorphisms was introduced in [19]. Many authors have developed this theory studying for example Fredholm, Weyl, Browder and Riesz Banach algebra element relative to homomorphisms, see for example [19, 5, 20, 21, 22, 41, 17, 42, 15, 23, 24, 2, 55, 43, 56, 57].

The main objective of this article is to characterize isolated spectral points and Koliha-Drazin invertible elements in quotient Banach algebras or in ranges of (not necessarily continuous) homomorphism between Banach algebras. In fact, in section 3, after having recalled some preliminary definitions and facts in section 2, given two complex unital Banach algebras \mathcal{A} and \mathcal{B} , an algebra homomorphism $\mathcal{T}: \mathcal{A} \rightarrow \mathcal{B}$, $a \in \mathcal{A}$ and $b = \mathcal{T}(a)$, poles and isolated spectral points of $b \in \mathcal{B}$ will be characterized in terms of Fredholm and Riesz elements relative to \mathcal{T} . In particular, Drazin and Koliha-Drazin invertible elements in the range of \mathcal{T} will be characterized

(see Theorem 3.2 and Corollary 3.3). It is worth noticing that the problem of lifting idempotents will be central in the proof of the aforementioned results. This problem has been considered by many authors and in general can not be solved, see for example [33, 14, 3, 4, 46, 15, 18, 30]. However, when the lifting is possible, for example when $\mathcal{T}: \mathcal{A} \rightarrow \mathcal{B}$ is surjective and has the Riesz property (see section 2), $\mathcal{B} = \mathcal{A}/\mathcal{J}$ ($\mathcal{J} \subset \mathcal{A}$ is the radical or a closed inessential ideal and \mathcal{T} is the quotient map $\pi: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{J}$), \mathcal{A} is a von Neumann or a real rank zero C^* -algebra, and $\mathcal{B} = \mathcal{A}/\mathcal{J}$ (\mathcal{J} a closed two sided ideal and $\mathcal{T} = \pi$), or \mathcal{A} is a unital C^* -algebra, $\mathcal{B} = \mathcal{A}/\mathcal{J}$ and \mathcal{J} is a closed two sided ideal that is an AW^* algebra or an AF C^* -algebra (see Remark 3.4), then poles, isolated spectral points, group, Drazin and Koliha-Drazin invertible elements in \mathcal{B} will be fully characterized using Fredholm and Riesz elements relative to \mathcal{T} or π .

In addition, in the Banach space operator context, the following characterization will be proved (compare with [6]). Let \mathcal{X} be a Banach space, $T \in \mathcal{L}(\mathcal{X})$ and $\pi(T) \in \mathcal{C}(\mathcal{X})$. Then, $\lambda \in \text{iso } \sigma(\pi(T))$ (respectively λ is a pole of $\pi(T)$) if and only if there exists an idempotent $P \in \mathcal{L}(\mathcal{X})$, $P \notin \mathcal{K}(\mathcal{X})$, such that if $\mathcal{X}_1 = P(\mathcal{X})$ and $\mathcal{X}_2 = (I - P)(\mathcal{X})$, then there are $T_1 \in \mathcal{L}(\mathcal{X}_1)$ a Riesz operator (respectively a power compact operator), $T_2 \in \mathcal{L}(\mathcal{X}_2)$ a Fredholm operator and $K \in \mathcal{K}(\mathcal{X})$ such that $T - \lambda = T_1 \oplus T_2 + K$, where $\mathcal{K}(\mathcal{X})$ is the closed ideal of compact operators. Naturally, the case $\lambda = 0$ provides a characterization of Koliha-Drazin and Drazin invertible elements in the Calkin algebra $\mathcal{C}(\mathcal{X})$, respectively. In the frame of arbitrary Hilbert space operators and for isolated points of the spectrum (respectively poles) of Calkin algebra elements, it will be proved that the operator T_1 can be chosen as quasi-nilpotent (respectively nilpotent). This last result will be also formulated in C^* -algebras. Finally, as an application of the main results, Drazin and Koliha-Drazin invertible Calkin algebra elements will be described in terms of the cosets of operator classes.

2. Preliminary definitions and facts

From now on \mathcal{A} will denote a complex unital Banach algebra with identity 1. Let \mathcal{A}^{-1} denote the set of all invertible elements in \mathcal{A} and \mathcal{A}^\bullet the set of all idempotents in \mathcal{A} . Given $a \in \mathcal{A}$, $\sigma(a)$, $\rho(a) = \mathbb{C} \setminus \sigma(a)$ and $\text{iso } \sigma(a)$ will stand for the spectrum, the resolvent set and the set of isolated spectral points of $a \in \mathcal{A}$, respectively. Recall that if $K \subseteq \mathbb{C}$, then $\text{acc } K$ is the set of limit points of K and $\text{iso } K = K \setminus \text{acc } K$.

As in the case of the spectrum of a Banach space operator, the resolvent function of $a \in \mathcal{A}$, $R(\cdot, a): \rho(a) \rightarrow \mathcal{A}$, is holomorphic and $\text{iso } \sigma(a)$ is the set of isolated singularities of $R(\cdot, a)$. Furthermore, as in the case of an operator, see [52, p. 305], if $\lambda_0 \in \text{iso } \sigma(a)$, then it is possible to consider the Laurent expansion of $R(\cdot, a)$ in terms of $(\lambda - \lambda_0)$. In fact,

$$R(\lambda, a) = \sum_{n \geq 0} a_n (\lambda - \lambda_0)^n + \sum_{n \geq 1} b_n (\lambda - \lambda_0)^{-n},$$

where a_n and b_n belong to \mathcal{A} and they are obtained in a standard way using the functional calculus. In addition, this representation is valid when $0 < |\lambda - \lambda_0| < \delta$, for any δ such that $\sigma(a) \setminus \{\lambda_0\}$ lies outside the circle $|\lambda - \lambda_0| = \delta$. What is more important, the discussion of [52, pp. 305 and 306] can be repeated for elements in a unital Banach algebra. Consequently, λ_0 will be called a *pole of order p of $R(\cdot, a)$* , if there exists $p \geq 1$ such that $b_p \neq 0$ and $b_m = 0$, for all $m \geq p + 1$. The set of poles of $a \in \mathcal{A}$ will be denoted by $\Pi(a) \subseteq \text{iso } \sigma(a)$. Define $I(a) = \text{iso } \sigma(a) \setminus \Pi(a)$.

Let \mathcal{X} be a Banach space and $\mathcal{L}(\mathcal{X})$ denote the algebra of all bounded and linear maps defined on and with values in \mathcal{X} . If $T \in \mathcal{L}(\mathcal{X})$, then $N(T)$ and $R(T)$ will stand for the null space and the range of T respectively. Note that $I \in \mathcal{L}(\mathcal{X})$ will stand for the identity map. Recall that the *descent* and the *ascent* of $T \in \mathcal{L}(\mathcal{X})$ are $d(T) = \inf\{n \geq 0: R(T^n) = R(T^{n+1})\}$ and $a(T) = \inf\{n \geq 0: N(T^n) = N(T^{n+1})\}$ respectively, where if some of the above sets is empty,

its infimum is then defined as ∞ , see for example [27]. Now well, according to [13, Lemma 3.4.2], $\Pi(T) = \{\lambda \in \sigma(a) : a(T - \lambda) \text{ and } d(T - \lambda) \text{ are finite}\}$. What is more, if \mathcal{A} is a unital Banach algebra and $a \in \mathcal{A}$, then, according to [10, Theorem 11], $\Pi(a) = \Pi(L_a) = \Pi(R_a)$, where $L_a \in \mathcal{L}(\mathcal{A})$ and $R_a \in \mathcal{L}(\mathcal{A})$ are the operators defined by left and right multiplication respectively, i.e., given $x \in \mathcal{A}$, $L_a(x) = ax$ and $R_a(x) = xa$.

Recall that $a \in \mathcal{A}$ is said to be *Drazin invertible*, if there exists a necessarily unique $b \in \mathcal{A}$ and some $m \in \mathbb{N}$ such that

$$ab = ba, \quad bab = b, \quad a^m ba = a^m.$$

If the Drazin inverse of a exists, then it will be denoted by a^d . In addition, the *index* of a , which will be denoted by $\text{ind}(a)$, is the least non-negative integer m for which the above equations hold. When $\text{ind}(a) = 1$, a will be said to be *group invertible*, and in this case its Drazin inverse will be referred as the group inverse of a . The set of Drazin invertible elements of \mathcal{A} will be denoted by \mathcal{A}^D ; see [16, 27, 50].

In addition, in [9] the *Drazin spectrum* of $a \in \mathcal{A}$ was introduced, i.e., $\sigma_D(a) = \{\lambda \in \mathbb{C} : a - \lambda \text{ is not Drazin invertible}\}$. Recall that according to [10, Theorem 12], see also [32, Proposition 1.5], $\sigma(a) = \sigma_D(a) \cup \Pi(a)$, $\sigma_D(a) \cap \Pi(a) = \emptyset$ and $\sigma_D(a) = \text{acc } \sigma(a) \cup I(a)$. In particular, if $\rho_D(a) = \mathbb{C} \setminus \sigma_D(a)$, then $\Pi(a) = \sigma(a) \cap \rho_D(a)$, equivalently, $a \in \mathcal{A}$ is Drazin invertible but not invertible if and only if $0 \in \Pi(a)$; see also [13, Lemma 3.4.2] and [27, Theorem 4]. Concerning the Drazin spectrum, see [9, 32, 10].

An element $a \in \mathcal{A}$ is said to be *generalized Drazin* or *Koliha-Drazin invertible*, if there exists $b \in \mathcal{A}$ such that

$$ab = ba, \quad bab = b, \quad aba = a + w, \quad (w \in \mathcal{A}^{qnil}),$$

where \mathcal{A}^{qnil} is the set of all quasinilpotent elements of \mathcal{A} . The Koliha-Drazin inverse is unique, if it exists, and it will be denoted by a^D ; see [28, 32, 47, 48]. Note that if $a \in \mathcal{A}$ is Drazin invertible, then a is generalized Drazin invertible. In fact, if $k = \text{ind}(a)$, then $(aba - a)^k = 0$. In addition, the Koliha-Drazin spectrum was introduced and studied in [32], i.e., $\sigma_{KD}(a) = \{\lambda \in \mathbb{C} : a - \lambda \text{ is not Koliha-Drazin invertible}\}$. According to [32, Proposition 1.5] (see also [28, Theorem 4.2]), $\sigma_{KD}(a) = \text{acc } \sigma(a)$. In particular, necessary and sufficient for $a \in \mathcal{A}$ to be Koliha-Drazin invertible but not invertible is that $0 \in \text{iso } \sigma(a)$. What is more, $a \in \mathcal{A}$ is generalized Drazin invertible but not Drazin invertible if and only if $a \in I(\mathcal{A})$. The set of Koliha-Drazin invertible elements of \mathcal{A} will be denoted by \mathcal{A}^{KD} .

In the following Remark characterizations of the elements in $\Pi(a)$ and $I(a)$, $a \in \mathcal{A}$, will be recalled.

Remark 2.1. Let A be a unital Banach algebra and consider $a \in A$. Recall that if $0 \neq p = p^2 \in A$, then pAp is a unital Banach algebra with unit p .

(i) Let $\lambda \in \mathbb{C}$. Then, $\lambda \in \text{iso } \sigma(a)$ if and only if there is $0 \neq p^2 = p \in A$ such that $ap = pa$, $(a - \lambda)p$ is quasi-nilpotent and $(a - \lambda + p) \in A^{-1}$.

(ii) Necessary and sufficient for $\lambda \in \mathbb{C}$ to belong to $\Pi(a)$ is that there exists $0 \neq p^2 = p \in A$ such that $ap = pa$, $(a - \lambda)p$ is nilpotent and $(a - \lambda + p) \in A^{-1}$. Note in particular that $((a - \lambda)p)^l = 0$ if and only if $l \geq k = \text{ind}(a - \lambda)$.

(iii) The number $\lambda \in \mathbb{C}$ belongs to $I(a)$ if and only if there is $0 \neq p^2 = p \in A$ such that $ap = pa$, $(a - \lambda)p$ is quasi-nilpotent but not nilpotent and $(a - \lambda + p) \in A^{-1}$.

Statements (i)-(ii) are well known, see for example [50, Proposition 1(a)], [28, Theorem 4.2] and [29, Theorem 1.2]. Statement (iii) is a direct consequence of statements (i) and (ii). In addition, $p \in \mathcal{A}^\bullet$ is the *spectral idempotent of a corresponding to λ* . To learn more results on isolated spectral points, see for example [35, 51].

Let \mathcal{X} be an infinite dimensional complex Banach space and denote by $\mathcal{K}(\mathcal{X}) \subset \mathcal{L}(\mathcal{X})$ the closed ideal of compact operators. Consider $\mathcal{C}(\mathcal{X})$ the Calkin algebra over X , i.e., the quotient

algebra $\mathcal{C}(\mathcal{X}) = \mathbf{L}(\mathcal{X})\mathcal{K}(\mathcal{X})$. Recall that $\mathcal{C}(X)$ is itself a Banach algebra with the quotient norm. Let

$$\pi: \mathbf{L}(\mathcal{X}) \rightarrow \mathcal{C}(\mathcal{X})$$

denote the quotient map.

Recall that $T \in \mathbf{L}(\mathcal{X})$ is said to be a *semi-Fredholm operator*, if $R(T)$ is closed and $\alpha(T) = \dim N(T)$ or $\beta(T) = \dim X/R(T)$ is finite, while if $\alpha(T)$ and $\beta(T)$ are both finite, T is said to be *Fredholm*. Denote by $\Phi(\mathcal{X})$ the set of all Fredholm operators defined on \mathcal{X} . It is well known that $\Phi(\mathcal{X})$ is a multiplicative open semigroup in $\mathbf{L}(\mathcal{X})$ and that

$$\Phi(\mathcal{X}) = \pi^{-1}(\mathcal{C}(\mathcal{X})^{-1}).$$

Given $T \in \mathbf{L}(\mathcal{X})$, T will be said to be *power compact*, if there exists $n \in \mathbb{N}$ such that $T^n \in \mathcal{K}(\mathcal{X})$, equivalently, if $\pi(T) \in \mathcal{C}(\mathcal{X})$ is nilpotent. The set of all power compact operators defined on \mathcal{X} will be denoted by $\mathcal{PK}(\mathcal{X})$. In addition, $T \in \mathbf{L}(\mathcal{X})$ will be said to be a *Riesz operator*, if $\pi(T) \in \mathcal{C}(\mathcal{X})$ is quasi-nilpotent, equivalently $\sigma_e(T) = \sigma(\pi(T)) = \{0\}$, where $\sigma_e(T)$ denotes the Fredholm spectrum of T (see [53, 54] and [13, Chapter 3]). Let $\mathcal{R}(\mathcal{X})$ denote the set of Riesz operators defined on the Banach space \mathcal{X} . It is clear that $\mathcal{PK}(\mathcal{X}) \subseteq \mathcal{R}(\mathcal{X})$.

Recall that according to the proof of [46, Theorem 1],

$$\pi^{-1}(\mathcal{C}(\mathcal{X})^\bullet) = \mathbf{L}(\mathcal{X})^\bullet + \mathcal{K}(\mathcal{X}).$$

Atkinson's theorem motivated the development of an abstract Fredholm theory in arbitrary complex unital Banach algebras with respect to an *inessential ideal*, i.e., a two sided ideal $\mathcal{J} \subseteq \mathcal{A}$ such that for every $z \in \mathcal{J}$, $\sigma(z)$ is either finite or is a sequence converging to zero. In fact, $x \in \mathcal{A}$ is said to be a *\mathcal{J} -Fredholm element* (or a *Fredholm element of \mathcal{A} relative to \mathcal{J}*), if there exists $y \in \mathcal{A}$ such that $xy - 1 \in \mathcal{J}$ and $yx - 1 \in \mathcal{J}$. The set of all \mathcal{J} -Fredholm elements will be denoted by $\Phi_{\mathcal{J}}(\mathcal{A})$. In addition, given $\mathcal{K} \subset \mathcal{A}$ a closed proper two sided ideal of \mathcal{A} , $x \in \mathcal{A}$ will be said to be a *Riesz element of \mathcal{A} relative to \mathcal{K}* , if $\pi(x) \in \mathcal{A}/\mathcal{K}$ is quasi-nilpotent, where $\pi: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{K}$ is the quotient map. The set of Riesz elements of \mathcal{A} relative to \mathcal{K} will be denoted $\mathcal{R}_{\mathcal{K}}(\mathcal{A})$. To learn the main results concerning the Fredholm and the Riesz theory in Banach algebras see [1, Chapter 5], [5] and the corresponding bibliography of these monographs. Recall that, according to [4, Lemma 1], when the inessential ideal \mathcal{J} is closed,

$$\pi^{-1}((\mathcal{A}/\mathcal{J})^\bullet) = \mathcal{A}^\bullet + \mathcal{J}.$$

Atkinson's theorem also motivated the Fredholm theory relative to homomorphisms between two Banach algebras. In fact, let \mathcal{A} and \mathcal{B} be two unital Banach algebras and consider $\mathcal{T}: \mathcal{A} \rightarrow \mathcal{B}$ an algebra homomorphism, not necessarily continue. Then, $\Phi_{\mathcal{T}}(\mathcal{A}) = \{a \in \mathcal{A}: \mathcal{T}(a) \in \mathcal{B}^{-1}\}$ is the set of Fredholm elements relative to the homomorphism \mathcal{T} , see [19]. Moreover, Browder, Weyl and other classes of elements relative to \mathcal{T} have been also introduced and studied, see for example [19, 20, 21, 22, 41, 17, 42, 15, 23, 24, 2, 55, 43, 56, 57]. To prove the main results of this article, it is important to recall another class of elements. Let $\mathcal{R}_{\mathcal{T}}(\mathcal{A}) = \{a \in \mathcal{A}: \mathcal{T}(a) \in \mathcal{B}^{qnil}\}$ be the set of Riesz elements relative to the homomorphism \mathcal{T} , see [56, 57]. A subset of this class is the set of \mathcal{T} -nilpotent elements, i.e., $\mathcal{N}_{\mathcal{T}}(\mathcal{A}) = \{a \in \mathcal{A}: \text{there exists } k \in \mathbb{N} \text{ such that } a^k \in N(\mathcal{T})\}$. Clearly, $\mathcal{N}_{\mathcal{T}}(\mathcal{A}) \subseteq \mathcal{R}_{\mathcal{T}}(\mathcal{A})$.

Furthermore, [46, Theorem 1] was generalized to Banach algebras. In fact, consider \mathcal{A} and \mathcal{B} two complex unital Banach algebras and $\mathcal{T}: \mathcal{A} \rightarrow \mathcal{B}$ an algebra homomorphism. It will be said that \mathcal{T} has the *Riesz property*, if $N(\mathcal{T})$ is an inessential ideal. Then, according to [15, Lemma 2], if $\mathcal{T}: \mathcal{A} \rightarrow \mathcal{B}$ has the Riesz property,

$$\mathcal{T}^{-1}(\mathcal{B}^\bullet) = \mathcal{A}^\bullet + N(\mathcal{T}).$$

Note that when \mathcal{A} is a Banach algebra and $\mathcal{J} \subset \mathcal{A}$ is a closed inessential ideal, then the quotient map $\pi: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{J}$ is a surjective Banach algebra homomorphism that has the Riesz property. Moreover, $\Phi_\pi(\mathcal{A}) = \Phi_{\mathcal{J}}(\mathcal{A})$ and $\mathcal{R}_\pi(\mathcal{A}) = \mathcal{R}_K(\mathcal{A})$. What is more, $\mathcal{N}_\pi(\mathcal{A}) = \{a \in \mathcal{A}: \text{there exists } k \in \mathbb{N} \text{ such that } a^k \in \mathcal{J}\}$. On the other hand, if \mathcal{A} and \mathcal{B} are two Banach algebras and $\mathcal{T}: \mathcal{A} \rightarrow \mathcal{B}$ is a surjective Banach algebra homomorphism that has the Riesz property, then $\mathcal{J} = N(\mathcal{T})$ is a closed inessential ideal of \mathcal{A} and it is not difficult to prove that $\Phi_{\mathcal{T}}(\mathcal{A}) = \Phi_{\mathcal{J}}(\mathcal{A})$ and $\mathcal{R}_{\mathcal{T}}(\mathcal{A}) = \mathcal{R}_{\mathcal{J}}(\mathcal{A})$.

To fully characterize Drazin and Koliha-Drazin invertible Calkin algebra elements, some classes of operators need to be considered.

Recall that an operator $T \in L(\mathcal{X})$ is said to be *semi-regular*, if $R(T)$ is closed and $N(T^n) \subseteq R(T)$, for all $n \in \mathbb{N}$, see for example [39, 40, 44, 38]. In addition, in [26] it was proved that given a semi-Fredholm operator $T \in L(\mathcal{X})$, there exist M and N two closed subspaces of \mathcal{X} invariant for T such that $\mathcal{X} = M \oplus N$, $T|_M$ is nilpotent and $T|_N$ is semi-regular. This decomposition is known as the *Kato decomposition*, and the operators satisfying these conditions, which were characterized in [31], are said to be *the quasi-Fredholm operators*, see [31, 38, 45].

On the other hand, an operator $T \in L(\mathcal{X})$ is said to be *B-Fredholm*, if there is $n \in \mathbb{N}$ such that $R(T^n)$ is closed and $T|_{R(T^n)} \in L(R(T^n))$ is Fredholm, see [7, 8]. In addition, according to [7, Proposition 2.6], a B-Fredholm operator is quasi-Fredholm; what is more, according to [7, Theorem 2.7], if $T \in L(\mathcal{X})$ is B-Fredholm, then there exist M and N two closed subspaces of \mathcal{X} invariant for T such that $\mathcal{X} = M \oplus N$, $T|_M$ is nilpotent and $T|_N$ is Fredholm (see also [45, Theorem 7]). The class of B-Fredholm operators defined on the Banach space \mathcal{X} will be denoted by $\mathcal{BF}(\mathcal{X})$.

Quasi-Fredholm operators were generalized to pseudo-Fredholm operators. In fact, given $T \in L(\mathcal{X})$, T will be said to be a *pseudo-Fredholm operator*, if there exist M and N two closed spaces of \mathcal{X} invariant for T such that $\mathcal{X} = M \oplus N$, $T|_M$ is quasi-nilpotent and $T|_N$ is semi-regular. This decomposition is called the *generalized Kato decomposition*, see [34, 35, 36, 37, 11]. In addition, an operator $T \in L(\mathcal{X})$ will be said to be *pseudo B-Fredholm*, if there are two closed subspaces of \mathcal{X} invariant for T such that $\mathcal{X} = M \oplus N$, $T|_M$ is quasi-nilpotent and $T|_N$ is Fredholm. Let $\mathcal{PBF}(\mathcal{X})$ denote the class of pseudo B-Fredholm operators defined on \mathcal{X} . It is clear that a pseudo B-Fredholm operator is pseudo-Fredholm.

Recall that according to [27, Theorem 4], if $T \in L(\mathcal{X})$ is Drazin invertible, then there exist two closed subspaces of \mathcal{X} invariant for T such that $\mathcal{X} = M \oplus N$, $T|_M$ is nilpotent and $T|_N$ is invertible. In particular, a Drazin invertible operator is B-Fredholm and B-Fredholm operators not only generalize Fredholm operators, but also Drazin invertible operators. In the next section it will be proved that the set of all cosets of B-Fredholm operators in a Calkin algebra on an arbitrary Hilbert space coincides with the set of all Drazin invertible elements of the Calkin algebra. Moreover, the set of all Koliha-Drazin invertible elements of the same type of Calkin algebra will be described using the class of pseudo B-Fredholm operator.

3. Main results

To prove the main results of this work, first some facts need to be recalled.

Remark 3.1. Let \mathcal{A} be a unital Banach algebra and consider $a \in \mathcal{A}$. If $0 \neq p = p^2 \in \mathcal{A}$, then define $p' = 1 - p$.

(i) Note that $a = pap + pap' + p'ap + p'ap'$.

(ii) Suppose that $ap = pa$, equivalently, $pa(1-p) = (1-p)ap = 0$. Then, it is not difficult to prove that necessary and sufficient for a to be invertible is that $pap \in (p\mathcal{A}p)^{-1}$ and $p'ap' \in (p'\mathcal{A}p')^{-1}$.

(iii) Let \mathcal{B} another unital Banach algebra and consider $\mathcal{T}: \mathcal{A} \rightarrow \mathcal{B}$ an algebra homomorphism. Suppose that $p \notin N(\mathcal{T})$. Then, if $q = T(p)$ and $\mathcal{T}_{p,q}: p\mathcal{A}p \rightarrow q\mathcal{B}q$, $0 \neq q = q^2$ and $\mathcal{T}_{p,q}$

is an algebra homomorphism between the unital Banach algebras $p\mathcal{A}p$ and $q\mathcal{B}q$. What is more, if \mathcal{T} is surjective, then $\mathcal{T}_{p,q}$ is surjective.

Next the main result will be presented. Note that the notation introduced in Remark 3.1 will be used in this section.

Theorem 3.2. *Let \mathcal{A} and \mathcal{B} be two unital Banach algebras and consider an algebra homomorphism $\mathcal{T}: \mathcal{A} \rightarrow \mathcal{B}$. Let $\lambda \in \mathbb{C}$ and $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $\mathcal{T}(a) = b$. Then, the following statements hold.*

- (i) *Suppose that there exists an idempotent $p \in \mathcal{A}$, $p \notin N(\mathcal{T})$, such that $p(a - \lambda)(1 - p)$ and $(1 - p)(a - \lambda)p \in N(\mathcal{T})$, $(1 - p)(a - \lambda)(1 - p) \in \Phi_{\mathcal{T}_{p',q'}}(p'\mathcal{A}p')$ and $p(a - \lambda)p \in \mathcal{R}_{\mathcal{T}}(\mathcal{A})$ (respectively $p(a - \lambda)p \in \mathcal{N}_{\mathcal{T}}(\mathcal{A})$, $p(a - \lambda)p \in \mathcal{R}_{\mathcal{T}}(\mathcal{A}) \setminus \mathcal{N}_{\mathcal{T}}(\mathcal{A})$). Then, $\lambda \in \text{iso } \sigma(b)$ (respectively $\lambda \in \Pi(b)$, $\lambda \in I(b)$). In particular, $0 \neq q = \mathcal{T}(p)$ is the spectral idempotent of b corresponding to λ .*
- (ii) *Let $\lambda \in \text{iso } \sigma(b)$ (respectively $\lambda \in \Pi(b)$, $\lambda \in I(b)$) and let $0 \neq q \in \mathcal{B}$ the spectral idempotent of b corresponding to λ . If there exists $p \in \mathcal{A}^\bullet$ such that $\mathcal{T}(p) = q$, then $p \notin N(\mathcal{T})$, $p(a - \lambda)(1 - p)$ and $(1 - p)(a - \lambda)p \in N(\mathcal{T})$, $(1 - p)(a - \lambda)(1 - p) \in \Phi_{\mathcal{T}_{p',q'}}(p'\mathcal{A}p')$ and $p(a - \lambda)p \in \mathcal{R}_{\mathcal{T}}(\mathcal{A})$ (respectively $p(a - \lambda)p \in \mathcal{N}_{\mathcal{T}}(\mathcal{A})$, $p(a - \lambda)p \in \mathcal{R}_{\mathcal{T}}(\mathcal{A}) \setminus \mathcal{N}_{\mathcal{T}}(\mathcal{A})$).*

Proof. (i). Suppose that there exists an idempotent $p \in \mathcal{A}$, $p \notin N(\mathcal{T})$, such that $p(a - \lambda)(1 - p)$ and $(1 - p)(a - \lambda)p \in N(\mathcal{T})$, $p(a - \lambda)p \in \mathcal{R}_{\mathcal{T}}(\mathcal{A})$ and $(1 - p)(a - \lambda)(1 - p) \in \Phi_{\mathcal{T}_{p',q'}}(p'\mathcal{A}p')$. Clearly, $q = \mathcal{T}(p)$ is an idempotent such that $0 \neq q$ and $qb = bq$. In addition, since $(a - \lambda)p = p(a - \lambda)p + (1 - p)(a - \lambda)p$, $(1 - p)(a - \lambda)p \in N(\mathcal{T})$ and $p(a - \lambda)p \in \mathcal{R}_{\mathcal{T}}(\mathcal{A})$, $(a - \lambda)p \in \mathcal{R}_{\mathcal{T}}(\mathcal{A})$, which in turn implies that $(b - \lambda)q \in \mathcal{B}^{qnil}$. Moreover, it is clear that $(1 - q)(b - \lambda)(1 - q) \in (q'\mathcal{B}q')^{-1}(\mathcal{T}_{p',q'}: p'\mathcal{A}p' \rightarrow q'\mathcal{B}q')$ as in Remark 3.1(iii) ($p' = 1 - p$ and $q' = 1 - q$). Now well, note that $(b - \lambda + q) = q(b - \lambda)q + q + (1 - q)(b - \lambda)(1 - q)$ and $q(b - \lambda + q)q = q(b - \lambda)q + q = (b - \lambda)q + q$. However, since $(b - \lambda)q \in \mathcal{B}^{qnil}$, $((b - \lambda)q + q) \in (q\mathcal{B}q)^{-1}$. Therefore, according to Remark 3.1(ii), $(b - \lambda + q) \in \mathcal{B}^{-1}$, which, according to Remark 2.1(i), implies that $\lambda \in \text{iso } \sigma(b)$.

If $p(a - \lambda)p \in \mathcal{N}_{\mathcal{T}}(\mathcal{A})$, then proceed as in the previous paragraph and note that since $p(a - \lambda)p \in \mathcal{N}_{\mathcal{T}}(\mathcal{A})$, then $q(b - \lambda)q = (b - \lambda)q \in \mathcal{B}$ is nilpotent. The remaining part of the implication follows as for the case $p(a - \lambda)p \in \mathcal{R}_{\mathcal{T}}(\mathcal{A})$ using in particular Remark 2.1(ii) instead of Remark 2.1(i).

The case $p(a - \lambda)p \in \mathcal{R}_{\mathcal{T}}(\mathcal{A}) \setminus \mathcal{N}_{\mathcal{T}}(\mathcal{A})$ is a direct consequence of what has been proved.

- (ii). Suppose that $\lambda \in \text{iso } \sigma(b)$ and consider $0 \neq q = q^2 \in \mathcal{B}$ the spectral idempotent of b corresponding to λ . In particular, $qb = bq$, $(b - \lambda)q$ is quasi-nilpotent and $(b - \lambda + q) \in \mathcal{B}^{-1}$. Suppose that there exists $p = p^2 \in \mathcal{A}$ such that $\mathcal{T}(p) = q$. As a result, $p \notin N(\mathcal{T})$. Decompose $a \in \mathcal{A}$, $\mathcal{T}(a) = b$, as follows: $a = pap + pa(1 - p) + (1 - p)ap + (1 - p)a(1 - p)$. Note that since $qb = bq$ and \mathcal{T} is an algebra homomorphism, $p(a - \lambda)(1 - p)$ and $(1 - p)(a - \lambda)p \in N(\mathcal{T})$. Moreover, since $\mathcal{T}(p(a - \lambda)p) = q(b - \lambda)q = (b - \lambda)q \in \mathcal{B}^{qnil}$, $p(a - \lambda)p \in \mathcal{R}_{\mathcal{T}}(\mathcal{A})$. Next consider $\mathcal{T}_{p',q'}: p'\mathcal{A}p' \rightarrow q'\mathcal{B}q'$. Since $(b - \lambda + q) \in \mathcal{B}^{-1}$, it is not difficult to prove that $(1 - q)(b - \lambda)(1 - q) \in (q'\mathcal{B}q')^{-1}$ (Remark 3.1(ii)). Then, clearly, $(1 - p)(a - \lambda)(1 - p) \in \Phi_{\mathcal{T}_{p',q'}}(p'\mathcal{A}p')$.

Suppose that $\lambda \in \Pi(b)$ and proceed as in the previous paragraph. Note that since $(q(b - \lambda)q)^k = ((b - \lambda)q)^k = 0$, $k = \text{ind } (b - \lambda)$, $((a - \lambda)p)^k \in N(\mathcal{T})$. In particular, $(a - \lambda)p \in \mathcal{N}_{\mathcal{T}}(\mathcal{A})$. The remaining part of the implication follows as in the previous paragraph.

As in the proof of statement (i), the case $\lambda \in I(b)$ is a direct consequence of what has been proved. \square

Next Drazin and Koliha-Drazin Banach algebra invertible elements in homomorphism ranges will be characterized.

Corollary 3.3. *Let \mathcal{A} and \mathcal{B} be two unital Banach algebras and consider an algebra homomorphism $\mathcal{T}: \mathcal{A} \rightarrow \mathcal{B}$. Let $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $\mathcal{T}(a) = b$. Then, the following statements hold.*

- (i) Suppose that there exists an idempotent $p \in \mathcal{A}$, $p \notin N(\mathcal{T})$, such that $pa(1-p)$ and $(1-p)ap \in N(\mathcal{T})$, $(1-p)a(1-p) \in \Phi_{\mathcal{T}_{p',q'}}(p'\mathcal{A}p')$ and $pap \in \mathcal{R}_{\mathcal{T}}(\mathcal{A})$ (respectively $pap \in \mathcal{N}_{\mathcal{T}}(\mathcal{A})$, $pap \in \mathcal{R}_{\mathcal{T}}(\mathcal{A}) \setminus \mathcal{N}_{\mathcal{T}}(\mathcal{A})$). Then, $b \in \mathcal{B}$ is Koliha-Drazin invertible but not invertible (respectively b is Drazin invertible but not invertible, b is Koliha-Drazin invertible but not Drazin invertible).
- (ii) Suppose that b is Koliha-Drazin invertible but not invertible (respectively Drazin invertible but not invertible, Koliha-Drazin invertible but not Drazin invertible) and let $q \in B$ the spectral idempotent of b corresponding to 0. If there exists $p \in \mathcal{A}^\bullet$ such that $T(p) = q$, then $p \notin N(\mathcal{T})$, $pa(1-p)$ and $(1-p)ap \in N(\mathcal{T})$, $(1-p)a(1-p) \in \Phi_{\mathcal{T}_{p',q'}}(p'\mathcal{A}p')$ and $pap \in \mathcal{R}_{\mathcal{T}}(\mathcal{A})$ (respectively $pap \in \mathcal{N}_{\mathcal{T}}(\mathcal{A})$, $pap \in \mathcal{R}_{\mathcal{T}}(\mathcal{A}) \setminus \mathcal{N}_{\mathcal{T}}(\mathcal{A})$).

Proof. The Corollary can be easily deduced from Theorem 3.2 taking in consideration the following observations. Recall that $b \in \mathcal{B}$ is Koliha-Drazin invertible but not invertible (respectively Drazin invertible but not invertible) if and only if $0 \in \text{iso } \sigma(b)$ ([28, Theorem 4.2]) (respectively $0 \in \Pi(b)$ ([32, Proposition 1.5] or [10, Theorem 12(i)])). \square

Note that in Corollary 3.3 (ii), if b is group invertible, then according to Remark 2.1(ii), $pap \in N(\mathcal{T})$.

To fully characterize poles, isolated points and Drazin and Koliha-Drazin invertible elements in quotient Banach algebras or homomorphism ranges, it is necessary to be able to lift spectral idempotents. In the following remark some examples for which the statements in Theorem 3.2 and Corollary 3.3 are equivalent will be presented.

Remark 3.4. (i). Let \mathcal{A} be a unital Banach algebra and consider \mathcal{R} the radical of \mathcal{A} and $\pi: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{R}$ the quotient map. Then, according to [49, Theorem 2.3.9], any idempotent in \mathcal{A}/\mathcal{R} can be lifted to an idempotent in \mathcal{A} .

(ii) Let \mathcal{A} be a unital Banach algebra and consider $\mathcal{J} \subset \mathcal{A}$ a closed inessential ideal. Let $\pi: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{J}$. Then, according to [4, Lemma 1], $\pi^{-1}((\mathcal{A}/\mathcal{J})^\bullet) = \mathcal{A}^\bullet + \mathcal{J}$. In the case of a C^* -algebra, see also [3, Theorem 15]. Note that according to [49, Theorem 2.3.5], the radical of a Banach algebra \mathcal{A} is a closed inessential ideal.

(iii). Let \mathcal{A} and \mathcal{B} be two unital Banach algebras and consider $\mathcal{T}: \mathcal{A} \rightarrow \mathcal{B}$ a surjective algebra homomorphism such that \mathcal{T} has the Riesz property. Then, according to [15, Lemma 2], $\mathcal{T}^{-1}(B^\bullet) = \mathcal{A}^\bullet + N(\mathcal{T})$.

(iv) Let \mathcal{A} be a C^* -algebra and consider $\mathcal{J} \subseteq \mathcal{A}$ a closed two-sided ideal. Let $\pi: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{J}$ be the quotient map. Given $b \in \mathcal{A}/\mathcal{J}$ such that $b^2 = b$, according to [18, Corollary 3] and [18, Corollary 4], if \mathcal{A} has real rank zero or is a von Neumann algebra, then there exists $a \in \mathcal{A}$ such that $\pi(a) = b$ and $a^2 = a$. The same result was proved when \mathcal{A} is a C^* -algebra, $\pi: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{J}$ is the quotient map and the closed two sided ideal \mathcal{J} satisfies Condition (A) in [14, p. 24] (for example if \mathcal{J} is an AW^* algebra ([25]) or an $\text{AF } C^*$ -algebra ([12])), see [14, Theorem 1].

(v) Note that if \mathcal{A} is a Banach algebra and $\mathcal{K} \subset \mathcal{A}$ is a closed two sided ideal, if $b^2 - b \in \mathcal{K}$, in general there is no $a \in \mathcal{A}$, $a^2 = a$ such that $a - b \in \mathcal{K}$. In fact, consider $\mathcal{A} = C[0, 1]$ and $\mathcal{K} = \{f \in C[0, 1]: f(0) = f(1) = 0\}$ (see [18, p. 431]).

In the following theorem conditions that fully characterized Drazin and Koliha-Drazin invertible elements in quotient Banach algebras or in Banach algebras that can be presented as the range of a (not necessarily continuous) homomorphism will be given. To this end, the following notion will be first introduced. Let \mathcal{A} and \mathcal{B} two complex unital Banach algebras and consider $\mathcal{T}: \mathcal{A} \rightarrow \mathcal{B}$ a surjective algebra homomorphism. \mathcal{T} will be said to have the *lifting property*, if given $q \in \mathcal{B}^\bullet$, there is $p \in \mathcal{A}^\bullet$ such that $\mathcal{T}(p) = q$, equivalently, $\mathcal{T}^{-1}(\mathcal{B}^\bullet) = \mathcal{A}^\bullet + N(\mathcal{T})$. Note that the homomorphism considered in Remark 3.4(i)-(iv) have the lifting property.

Theorem 3.5. Let \mathcal{A} and \mathcal{B} be two unital Banach algebras and consider a surjective algebra homomorphism $\mathcal{T}: \mathcal{A} \rightarrow \mathcal{B}$ such that \mathcal{T} has the lifting property. Let $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $\mathcal{T}(a) = b$. Then, b is Koliha-Drazin invertible but not invertible (respectively Drazin invertible

but not invertible, Koliha-Drazin invertible but not Drazin invertible) if and only if there exist $p^2 = p \in \mathcal{A}$, $p \notin N(\mathcal{T})$, $x \in \mathcal{A}$, $pxp \in \mathcal{R}_{\mathcal{T}}(\mathcal{A})$ (respectively $pxp \in \mathcal{N}_{\mathcal{T}}(\mathcal{A})$, $pxp \in \mathcal{R}_{\mathcal{T}}(\mathcal{A}) \setminus \mathcal{N}_{\mathcal{T}}(\mathcal{A})$), $y \in (1-p)\mathcal{A}(1-p)$, $y \in \Phi_{\mathcal{T}_{p',q}}(p'\mathcal{A}p')$ ($q = \mathcal{T}(p)$) and $z \in N(\mathcal{T})$ such that $a = pxp + y + z$.

Proof. Apply Remark 2.1, Remark 3.1 and Theorem 3.2 or Corollary 3.3. \square

Remark 3.6. Under the same hypothesis and notation of Theorem 3.5, note that $b \in \mathcal{B}$ is group invertible if and only if $pap \in N(T)$ (Remark 2.1(ii)), so that, in this case, it is possible to chose $x = 0$. Moreover, when Theorem 3.5 holds, it is possible to consider $a = x$, $y = (1-p)a(1-p)$ and $z = pa(1-p) + (1-p)ap$ (Theorem 3.2 or Corollary 3.3). In addition, given $\lambda \in \mathbb{C}$, to characterize when $\lambda \in \text{iso } \sigma(b)$ (respectively $\lambda \in \Pi(b)$, $\lambda \in \text{iso } \sigma(b) \setminus \Pi(b)$), it is enough to consider when $b - \lambda$ is Koliha-Drazin invertible but not invertible (respectively Drazin invertible but not invertible, Koliha-Drazin invertible but not Drazin invertible), see the proof of Corollary 3.3.

Naturally, the main application of Theorem 3.2 and Corollary 3.3 is the Calkin algebra.

Theorem 3.7. Let \mathcal{X} be a complex Banach space and consider $T \in L(\mathcal{X})$, $\lambda \in \mathbb{C}$ and $\pi(T) \in \mathcal{C}(\mathcal{X})$. The following statements hold.

- (i) Necessary and sufficient for $\lambda \in \text{iso } \sigma(\pi(T))$ is that there exists an idempotent $P \in L(\mathcal{X})$, $P \notin \mathcal{K}(\mathcal{X})$, such that if $\mathcal{X}_1 = P(\mathcal{X})$ and $\mathcal{X}_2 = (I-P)(\mathcal{X})$, then there are $T_1 \in \mathcal{R}(\mathcal{X}_1)$, $T_2 \in \Phi(\mathcal{X}_2)$ and $K \in \mathcal{K}(\mathcal{X})$ such that $T - \lambda = T_1 \oplus T_2 + K$.
- (ii) The number $\lambda \in \mathbb{C}$ is a pole of $\pi(T)$ if and only if there exists an idempotent $P \in L(\mathcal{X})$, \mathcal{X}_1 , \mathcal{X}_2 , T_2 and K as in statement (i) and $T_1 \in \mathcal{PK}(\mathcal{X}_1)$, such that $T - \lambda = T_1 \oplus T_2 + K$.
- (iii) A necessary and sufficient condition for $\lambda \in I(\pi(T))$ is that there exists an idempotent $P \in L(\mathcal{X})$, \mathcal{X}_1 , \mathcal{X}_2 , T_2 and K as in statement (i) and $T_1 \in \mathcal{R}(\mathcal{X}_1) \setminus \mathcal{PK}(\mathcal{X}_1)$, such that $T - \lambda = T_1 \oplus T_2 + K$.

Proof. Adapt the proof of Theorem 3.2 to the case under consideration, using in particular [46, Theorem 1]. \square

Under the same hypothesis and notation in Theorem 3.7, to characterize when $\pi(T) \in \mathcal{C}(\mathcal{X})$ is Koliha-Drazin invertible but not invertible (respectively Drazin invertible but not invertible, Koliha-Drazin invertible but not Drazin invertible), consider the case $\lambda = 0$ (see the proof of Corollary 3.3). In addition, when $\pi(T) \in \mathcal{C}(\mathcal{X})$ is group invertible, according again to Remark 2.1(ii), note that $T_1 \in \mathcal{K}(\mathcal{X})$, so that, in this case, $T = 0 \oplus T_2 + K$.

Next the Hilbert space case will be consider. Compare with [6, Theorem 2.2]. Note that according to [30, Section 4], the structure theorem for polynomially compact operators proved in [33, Theorem 2.4] holds for arbitrary Hilbert spaces.

Corollary 3.8. Let \mathcal{H} be a Hilbert space and consider $T \in L(\mathcal{H})$ and $\pi(T) \in \mathcal{C}(\mathcal{H})$. The following statments hold.

- (i) $\lambda \in \text{iso } \sigma(\pi(T))$ if and only if there exists an idempotent $P \in L(\mathcal{H})$, $P \notin \mathcal{K}(\mathcal{H})$, such that if $\mathcal{H}_1 = P(\mathcal{H})$ and $\mathcal{H}_2 = (I-P)(\mathcal{H})$, then there are a quasi-nilpotent operator $T_1 \in L(\mathcal{H}_1)$, $T_2 \in \Phi(\mathcal{H}_2)$ and $K \in \mathcal{K}(\mathcal{H})$ such that $T - \lambda = T_1 \oplus T_2 + K$.
- (ii) $\lambda \in \Pi(\pi(T))$ if and only if there exists $P \in L(\mathcal{H})$, \mathcal{H}_1 , \mathcal{H}_2 and $T_2 \in L(\mathcal{H}_2)$ as in statement (i) and a nilpotent operator $T_1 \in L(\mathcal{H}_1)$ such that $T - \lambda = T_1 \oplus T_2 + K$.
- (iii) $\lambda \in I(\pi(T))$ if and only if there exists $P \in L(\mathcal{H})$, \mathcal{H}_1 and \mathcal{H}_2 , $T_2 \in L(\mathcal{H}_2)$ as in statement (i) and a quasi-nilpotent but not nilpotent operator $T_1 \in L(\mathcal{H}_1)$ such that $T - \lambda = T_1 \oplus T_2 + K$.

Proof. (i). According to Theorem 3.7(i), it is enough to prove the necessary condition. To this end, according to Theorem 3.7 (i), there exist an idempotent $P \in L(\mathcal{H})$, $\mathcal{H}_1 = R(P)$, $\mathcal{H}_2 = R(I-P)$, $T'_1 \in \mathcal{R}(\mathcal{H}_1)$, $T_2 \in \Phi(\mathcal{H}_2)$ and $K' \in \mathcal{K}(\mathcal{H})$ such that $T - \lambda = T'_1 \oplus T_2 + K'$.

However, according to the West decomposition of Riesz operators ([54, Theorem 7.5]), there is $K_1 \in \mathcal{K}(\mathcal{H}_1)$ such that $T'_1 + K_1$ is quasi-nilpotent. To conclude the proof, Define $T_1 = T'_1 + K_1$ and

$$K = K' + \begin{pmatrix} -K_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

- (ii) Proceed as in the proof of statement (i) but instead of [54, Theorem 7.5] use the structure theorem of polynomially compact operators of C. Olsen ([33, Theorem 2.4] and [30, Section 4]).
 (iii). It is a consequence of statements (i) and (ii). \square

In the same conditions of Corollary 3.8, to characterize when $\pi(T) \in \mathcal{C}(\mathcal{H})$ is Koliha-Drazin invertible but not invertible (respectively Drazin invertible but not invertible, Koliha-Drazin invertible but not Drazin invertible), it is enough to consider $\lambda = 0$ and statement (i) (respectively statement (ii), (iii)), see Corollary 3.3.

Corollary 3.8 can be formulated in C^* -algebras.

Proposition 3.9. *Let \mathcal{A} be a C^* -algebra and consider $\mathcal{J} \subset \mathcal{A}$ a closed inessential ideal. Let $a \in \mathcal{A}$ and $b \in \mathcal{A}/\mathcal{J}$ such that $b = \pi(a)$ ($\pi: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{J}$ the quotient map). Then b is Koliha-Drazin invertible but not invertible (respectively Drazin invertible but not invertible, Koliha-Drazin invertible but not Drazin invertible) if and only if there exist $p^2 = p \in \mathcal{A}$, $p \notin \mathcal{J}$, $x \in \mathcal{A}$ quasi-nilpotent (respectively nilpotent, quasi-nilpotent but not nilpotent) $y \in (1 - p)\mathcal{A}(1 - p)$, $y \in \Phi_{\pi_{p',q'}}(p'Ap')$ ($q = \pi(p)$), and $z \in \mathcal{J}$ such that $a = pxp + y + z$.*

Proof. Suppose that there exist $p^2 = p \in \mathcal{A}$, $p \notin \mathcal{J}$, $x \in \mathcal{A}^{qnil}$, $y \in (1 - p)\mathcal{A}(1 - p)$, $y \in \Phi_{\pi_{p',q'}}(p'Ap')$ and $z \in \mathcal{J}$ such that $a = pxp + y + z$. Then, according to [49, Theorem 1.6.15], $pxp \in (pAp)^{qnil} \subseteq \mathcal{R}_{\mathcal{J}}(\mathcal{A})$. Therefore, according to Theorem 3.5, $b \in \mathcal{A}/\mathcal{J}$ is Koliha-Drazin invertible.

On the other hand, if $b \in \mathcal{A}/\mathcal{J}$ is Koliha-Drazin invertible, then according to Theorem 3.5, there exist $p^2 = p \in \mathcal{A}$, $p \notin \mathcal{J}$, $x' \in \mathcal{A}$, $px'p \in \mathcal{R}_{\mathcal{J}}(\mathcal{A})$, $y \in (1 - p)\mathcal{A}(1 - p)$, $y \in \Phi_{\pi_{p',q'}}(p'Ap')$ and $z' \in \mathcal{J}$ such that $a = px'p + y + z'$. However, according to [5, Corollary $C^*.2.5$], there are s quasi-nilpotent and $t \in \mathcal{J}$ such that $px'p = s + t$ (the West decomposition in C^* -algebras). As a result, $px'p = psp + ptp$. Let $x = s$ and $z = z' + ptp$. Then, $a = pxp + y + z$, $x \in \mathcal{A}^{qnil}$, $y \in (1 - p)\mathcal{A}(1 - p)$, $y \in \Phi_{\pi_{p',q'}}(p'Ap')$ and $z \in \mathcal{J}$.

To prove the equivalence for the case $b \in \mathcal{A}/\mathcal{J}$ Drazin invertible but not invertible, proceed as before using [3, Theorem 15] instead of [5, Corollary $C^*.2.5$].

The case $b \in \mathcal{A}/\mathcal{J}$ Koliha-Drazin invertible but not Drazin invertible is a consequence of what has been proved. \square

Remark 3.10. Under the same hypothesis of Proposition 3.9, the case $\lambda \in \text{iso } \sigma(b)$ (respectively $\lambda \in \Pi(b)$, $\lambda \in I(b)$) can be characterized as in Remark 3.6. Note also that, according to [49, Theorem 1.6.15], if $x \in \mathcal{A}^{qnil}$, then $pxp \in (pAp)^{qnil}$.

On the other hand, a statement similar to the one in Proposition 3.9 can be considered for the case $\mathcal{T}: \mathcal{A} \rightarrow \mathcal{B}$, \mathcal{A} and \mathcal{B} two unital Banach algebras and \mathcal{T} a surjective Banach algebra homomorphism that has the Riesz property (see section 2).

Finally, Drazin and Koliha-Drazin invertible elements in Calkin algebras will be characterized using classes of operators. Recall that an Atkinson-type theorem for B-Fredholm operators holds. In fact, according to [9, Theorem 3.4], $T \in \mathbf{L}(\mathcal{X})$ is a B-Fredholm operator if and only if its coset in $\mathbf{L}(\mathcal{X})/F_0(\mathcal{X})$ is Drazin invertible, where $F_0(\mathcal{X})$ is the ideal of finite rank operators in $\mathbf{L}(\mathcal{X})$. What is more, in $\mathcal{C}(\mathcal{X})$ this characterization does not hold (see [9, Remark p. 256]). However, in the case of a Calkin algebra on a Hilbert space, Drazin invertible elements will be characterized using B-Fredholm operators. To consider the Banach space case, two classes of operators need to be introduced.

Let \mathcal{X} be a Banach space and consider $T \in \mathcal{L}(\mathcal{X})$. The operator T will be said to be *Riesz-Fredholm* (respectively *power compact-Fredholm*), if there exist M and N two closed subspaces of \mathcal{X} invariant for T such that $\mathcal{X} = M \oplus N$, $T|_M \in \mathcal{RF}(M)$ (respectively $T|_M \in \mathcal{PKF}(M)$) and $T|_N \in \Phi(N)$. These classes of operators will be denoted by $\mathcal{RF}(\mathcal{X})$ and $\mathcal{PKF}(\mathcal{X})$ respectively.

Theorem 3.11. *Let \mathcal{X} be a Banach space and consider $\pi: \mathcal{L}(\mathcal{X}) \rightarrow \mathcal{C}(\mathcal{X})$. The following statements hold.*

- (i) $\pi(\mathcal{RF}(\mathcal{X})) = \mathcal{C}(\mathcal{X})^{KD}$, equivalently, $\pi^{-1}(\mathcal{C}(\mathcal{X})^{KD}) = \mathcal{RF}(\mathcal{X}) + \mathcal{K}(\mathcal{X})$.
- (ii) $\pi(\mathcal{PKF}(\mathcal{X})) = \mathcal{C}(\mathcal{X})^D$, equivalently, $\pi^{-1}(\mathcal{C}(\mathcal{X})^D) = \mathcal{PKF}(\mathcal{X}) + \mathcal{K}(\mathcal{X})$.

Proof. Apply Theorem 3.7. \square

Theorem 3.12. *Let \mathcal{H} be a Banach space and consider $\pi: \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{C}(\mathcal{H})$. The following statements hold.*

- (i) $\pi(\mathcal{PBF}(\mathcal{H})) = \mathcal{C}(\mathcal{H})^{KD}$, equivalently, $\pi^{-1}(\mathcal{C}(\mathcal{H})^{KD}) = \mathcal{PBF}(\mathcal{H}) + \mathcal{K}(\mathcal{H})$.
- (ii) $\pi(\mathcal{BF}(\mathcal{H})) = \mathcal{C}(\mathcal{H})^D$, equivalently, $\pi^{-1}(\mathcal{C}(\mathcal{H})^D) = \mathcal{BF}(\mathcal{H}) + \mathcal{K}(\mathcal{H})$.

Proof. Apply Corollary 3.8. \square

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